`Comments, suggestions, corrections, and references are most welcomed!`

CAPABLE TWO-GENERATOR 2-GROUPS OF CLASS TWO

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ABSTRACT. A group is called capable if it is a central factor group. We characterize the capable 2-generator 2-groups of class 2 in terms of a standard presentation.

1. INTRODUCTION

Recall that a group G is said to be **capable** if there exists some group K such that $K/Z(K)$ is isomorphic to G. There are groups which are not capable (nontrivial cyclic groups being a well known example), and the question of which finite p-groups are capable plays an important role in their classification. See for example P. Hall's comments in his landmark [6] (middle of pp. 137).

In [2] Baer characterized the capable groups that are a direct sum of cyclic groups; the question of which groups are capable has received renewed attention thanks to results connecting the question to certain cohomological functors, most notably the nonabelian tensor square [4, 5]. There also has been work studying the restrictions that capability places on the structure of a group; see for example [7, 8].

More recently, Bacon and Kappe [1] characterized the capable 2-generator p-groups of class two, with p an odd prime, using the nonabelian tensor square. The author obtained the same result using other methods [10]. The purpose of the present note is to extend that characterization to the case of $p = 2$, yielding a characterization of capability for 2-generated finite groups of class at most two.

2. A NECESSARY CONDITION

Notation will be standard; we use the convention that the commutator of two elements is given by $[x, y] = x^{-1}y^{-1}xy$, and commutators will be written left-normed, so that $[x, y, z] = [[x, y], z]$, etc. We will also write x^y to denote $y^{-1}xy$.

The following commutator identities are well known, and may be verified by direct calculation:

Proposition 2.1. *Let G be any group. Then for all $x, y, z \in G$, $r, s, n \in \mathbb{Z}$,*

(a) $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$.
(b) $[x, yz] = [x, z][x, y]^z = [x, z][z, [y, x]][x, y]$.
(c) $[x^r, y^s] \equiv [x, y]^{rs}[x, y, x]^{s\binom{r}{2}}[x, y, y]^{r\binom{s}{2}} \pmod{G_4}$.
(d) $[y^s, x^r] \equiv [x, y]^{-rs}[x, y, x]^{-s\binom{r}{2}}[x, y, y]^{-r\binom{s}{2}} \pmod{G_4}$.
(e) $(xy)^n \equiv x^n y^n [y, x]^{\binom{n}{2}} \pmod{G_3}$.

Here, $\binom{m}{2} = \frac{m(m-1)}{2}$ for all integers m, and G_k is the k-th term of the lower central series of G.

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In [10], we established a necessary condition for capability of a finitely generated p-group of class k in terms of the orders of the elements of a minimal generating set. In our setting of 2-groups of class 2, the necessary condition becomes:

Proposition 2.2. *Let G be a group of class two, minimally generated by $x_1,\ldots,x_m$, with x_i of order 2^{r_i} and satisfying $1\leq r_1\leq r_2\leq\cdots\leq r_m$. If G is capable, then $m>1$ and $r_m\leq r_{m-1}+1$.*

When $r_m=r_{m-1}+1$, we can add a condition on the orders of the commutators $[r_m,r_i]$, $1\leq i\leq m-1$:

Theorem 2.3. *Let G be a group of class two, minimally generated by elements $x_1,\ldots,x_m$ of orders $1<2^{r_1}\leq\cdots\leq 2^{r_m}$, respectively, and assume that $m>1$ and $r_m=r_{m-1}+1$. If G is capable, then at least one of the commutators $[r_m,r_i]$ is of order $2^{r_{m-1}}$, for some $i\in\{1,\ldots,m-1\}$.*

The theorem will follow from the next lemma:

Lemma 2.4. *Let K be a nilpotent group of class three, let $y_1,\ldots,y_m$ be elements of K which generate K modulo $Z(K)$, and assume that $y_i^{2^{r_i}}\in Z(K)$, with*

$$1\leq r_1\leq\cdots\leq r_{m-1}\leq r_m.$$

If there exist integers $0\leq\gamma_i<r_{m-1}$, $i=1,\ldots,m-1$ such that $[y_m,y_i]^{2^{\gamma_i}}$ commutes with y_i and y_m, then $y_m^{2^{r(m-1)}}\in Z(K)$.

Proof. To avoid subscripts of exponents, let $\alpha=r_{m-1}$. We want to show that $y_m^{2^\alpha}$ commutes with y_i, $i=1,\ldots,m-1$. Since $[y_m,y_i]^{2^{\gamma_i}}\in Z(\langle y_i,y_m\rangle)$, it follows that

$$e=[[y_m,y_i]^{2^{\gamma_i}},y_i]=[y_m,y_i,y_i]^{2^{\gamma_i}}=[[y_m,y_i]^{2^{\gamma_i}},y_m]=[y_m,y_i,y_m]^{2^{\gamma_i}}.$$

Since $\gamma_i<\alpha$, $[y_m,y_i,y_i]^{2^{\alpha-1}}$ is a power of $[y_m,y_i,y_i]^{2^{\gamma_i}}$, and therefore is trivial. Same with $[y_m,y_i,y_m]^{2^{\alpha-1}}$. We also have

$$\begin{aligned} e &= [y_m,y_i^{2^\alpha}]=[y_m,y_i]^{2^\alpha}[y_m,y_i,y_i]^{\binom{2^\alpha}{2}}\\ &= [y_m,y_i]^{2^\alpha}[y_m,y_i,y_i]^{2^{\alpha-1}(2^\alpha-1)}=[y_m,y_i]^{2^\alpha}, \end{aligned}$$

so we conclude that $[y_m,y_i]^{2^\alpha}=e$ for $i=1,\ldots,m-1$. Therefore,

$$[y_m^{2^\alpha},y_i]=[y_m,y_i]^{2^\alpha}[y_m,y_i,y_m]^{\binom{2^\alpha}{2}}=e.$$

This proves that $y_m^{2^\alpha}\in Z(K)$, as claimed. □

The proof of Theorem 2.3 is now immediate:

Proof. Let K be a group such that $G\cong K/Z(K)$. Let $y_1,\ldots,y_m$ be elements of K that project onto $x_1,\ldots,x_m$, respectively. We know that for each i, the order of $[x_m,x_i]$ divides 2^{r_i}; if all commutators $[x_m,x_i]$ have order strictly less than $2^{r_{(m-1)}}$, then by Lemma 2.4 we must have $y_m^{r_{(m-1)}}\in Z(K)$, and therefore $r_m\leq r_{(m-1)}$. So if $r_m=r_{(m-1)}+1$, then we must have that at least one of the commutators $[x_m,x_i]$ is of order at least $2^{r_{(m-1)}}$; since this is the highest order it can have, we conclude that there exists some $i<m$ with $r_i=r_{m-1}$, and with $[x_m,x_i]$ of order exactly $2^{r_{(m-1)}}$, as claimed. □

3. The classification of two-generated 2-groups of class two

Since we aim to characterize the capable 2-generator 2-groups of class two, we need a description of these groups. We modify the presentation from [9] to facilitate our own analysis.

Theorem 3.1 (Theorem 2.5 in [9]). *Let G be a finite nonabelian 2-generator 2-group of nilpotency class two. Then G is isomorphic to exactly one group of the following types:*

(i) $G \cong \left\langle a, b \;\middle|\; a^{2^\alpha} = b^{2^\beta} = [a,b]^{2^\gamma} = [a,b,a] = [a,b,b] = e \right\rangle$, *where α, β, and γ are positive integers satisfying $\alpha \geq \beta \geq \gamma$.*

(ii) $G \cong \left\langle a, b \;\middle|\; a^{2^\alpha} = b^{2^\beta} = [a,b,a] = [a,b,b] = e, \quad a^{2^{\alpha+\sigma-\gamma}} = [a,b]^{2^\sigma} \right\rangle$, *with $\alpha, \beta, \gamma, \sigma$ integers satisfying $\beta \geq \gamma > \sigma \geq 0$, $\alpha + \sigma \geq 2\gamma$, and $\alpha + \beta + \sigma > 3$.*

(iii) $G \cong \left\langle a, b \;\middle|\; a^{2^{\gamma+1}} = b^{2^{\gamma+1}} = [a,b]^{2^\gamma} = [a,b,a] = [a,b,b] = e, a^{2^\gamma} = b^{2^\gamma} = [a,b]^{2^{\gamma-1}} \right\rangle$, *with γ a positive integer.*

Remark 3.2. Although in [9] the commutator convention is that $[x,y] = xyx^{-1}y^{-1}$, this difference is immaterial because in a nilpotent group of class two we always have $xyx^{-1}y^{-1} = x^{-1}y^{-1}xy$.

The condition $\alpha+\beta+\sigma > 3$ in type (ii) is to prevent the dihedral group of order 8 from appearing in both types (i) (with $\alpha = \beta = \gamma = 1$) and (ii) (with $\alpha = 2$, $\beta = \gamma = 1$, $\sigma = 0$).

Remark 3.3. In the original statement of Theorem 3.1 in [9], the authors split (ii) into two families, one with $\sigma = 0$ and one with $\sigma > 0$; the former family consists of the split metacyclic 2-groups of class two. We could even combine types (i) and (ii) into a single expression by using the presentation in (ii) and allowing $\gamma = \sigma > 0$. However, the condition necessary to guarantee uniqueness becomes more cumbersome, and since we will deal with groups of type (i) separately below anyway, we have kept the distinction.

I will refer to the three types informally as follows: groups of type (i) will be the *"coproduct type"* groups, since they arise as the coproduct of two cyclic groups in the subvariety of all nilpotent groups of class two in which 2^γ-powers are central. Groups of type (ii) will be the *"general type"* groups, since they give rise to all groups between split metacyclic and coproducts. When p is odd, the two-generator p-groups of class two are counterparts to these two families; the groups of type (iii) above have no such counterparts for odd primes, so groups of type (iii) will be called of *"exceptional type."*

4. The exceptional type case

The exceptional type is straightforward: a group of exceptional type is never capable.

Theorem 4.1. *If G is of exceptional type (that is, presented as in Theorem 3.1(iii)), then G is not capable.*

Proof. Let K be a group of class three, and assume that $K/N \cong G$, with N a central subgroup. We want to show that $N \neq Z(K)$. Let $x, y \in K$ project down to a and b in G, respectively. Since $a^{2^\gamma} = b^{2^\gamma}$ in G, we must have $x^{2^\gamma}y^{-2^\gamma} \in N \subset Z(K)$.

Since y^{2^γ} commutes with y, so does the product $(x^{2^\gamma}y^{-2^\gamma})y^{2^\gamma} = x^{2^\gamma}$. As x^{2^γ} also commutes with x, we conclude that it is in fact central in K. Since $a^{2^\gamma} \neq e$, we must have $x^{2^\gamma} \notin N$. This proves $N \neq Z(K)$, as desired. $\square$

5. Normal forms and the coproduct type case

Our method consists of constructing explicit witnesses to the capability of the given groups when appropriate. As we attempt the construction, the cases where the group is not capable will become evident by the appearance of undesired relations in the potential witness. In essence, we are constructing a simplified version of the "generalised extension of G;" see Theorem III.3.9 in [3]. Our starting point will be the 3-nilpotent product of cyclic groups (see [10–12] for details). We restrict the general definition to the case we are interested in:

Definition 5.1. Let $A_1,\ldots,A_t$ be cyclic groups. The 3-nilpotent product of the A_i, denoted $A_1 \amalg^{\mathfrak{N}_3} \cdots \amalg^{\mathfrak{N}_3} A_t$, is defined to be F/F_4, where F is the free product of the A_i, $F = A_1 * \cdots * A_t$, and F_4 is the fourth term of the lower central series of F.

We will mostly be concerned with the case $t = 2$, so we restrict our presentation below to that situation. The normal form given in [11] for the 3-nilpotent product of cyclic 2-groups seems to be difficult to use in most of our cases. That choice of normal form was made to facilitate the description of the multiplication table, and we will not need the multiplication table. We will therefore also provide an alternative normal form which we will use to study some of the cases below.

Theorem 5.2 (Struik, Theorem 4 in [11])**.** *Let a and b generate cyclic groups of order 2^α and 2^β, respectively, with $\alpha \geq \beta \geq 1$. Let $G = \langle a\rangle \amalg^{\mathfrak{N}_3} \langle b\rangle$ be their 3-nilpotent product. Then every element of G may be expressed uniquely in the form*

$$a^r b^s [a,b]^t [a^2,b]^u [a,b^2]^v \tag{5.3}$$

where r is unique modulo 2^α; s is unique modulo 2^β; t is unique modulo $2^{\beta+1}$; v is unique modulo $2^{\beta-1}$; and u is unique modulo 2^β if $\alpha \neq \beta$, and unique modulo $2^{\beta-1}$ if $\alpha = \beta$.

Corollary 5.4 (cf. Theorems 5.1 and 5.2 in [10])**.** *Let G be as above. The center of G is generated by $a^{2^{\beta+1}}$, $[a,b]^2[a^2,b]^{-1}$, and $[a,b]^2[a,b^2]^{-1}$. Therefore,*

$$G/Z(G) \cong \bigl\langle x,y \bigm| x^{2^{\max\{\alpha,\beta+1\}}} = y^{2^\beta} = [x,y]^{2^\beta} = [x,y,x] = [x,y,y] = e\bigr\rangle.$$

Suppose, however, that want to make $[a,b]^{2^\gamma}$ is central, for some $\gamma < \beta$, so the central quotient will have $[a,b]$ of order 2^γ. In that case, it is more convenient to switch to a normal form that uses basic commutators on a and b. We have the following:

Theorem 5.5. *Let $\langle a\rangle$, $\langle b\rangle$ be cyclic groups of order 2^α and 2^β, respectively, with $\alpha \geq \beta \geq 1$. Let $G = \langle a\rangle \amalg^{\mathfrak{N}_3} \langle b\rangle$, and let γ be an integer, $1 \leq \gamma < \beta$. Let N be the central subgroup of G generated by $[a,b,a]^{2^\gamma} = [a,b]^{-2^{\gamma+1}}[a^2,b]^{2^\gamma}$ and $[a,b,b]^{2^\gamma} = [a,b]^{-2^{\gamma+1}}[a,b^2]^{2^\gamma}$. Then every element of $K = G/N$ can be written uniquely as $k = a^r b^s [a,b]^t [a,b,a]^u [a,b,b]^v$ (abusing notation and writing a instead*

of aN, etc.), where r is unique modulo 2^α, s and t are unique modulo 2^β, and u and v are unique modulo 2^γ.

Proof. We may rewrite any element of G as

$$g = a^r b^s [a,b]^t [a,b,a]^u [a,b,b]^v \tag{5.6}$$

by using the normal form from Theorem 5.2, the identities $[a^2,b] = [a,b]^2[a,b,a]$ and $[a,b^2] = [a,b]^2[a,b,b]$, and the fact that $[G,G]$ is abelian. Note that:

$$e = [a,b^{2^\beta}] = [a,b]^{2^\beta}[a,b,b]^{\binom{2^\beta}{2}} = [a,b]^{2^\beta}[a,b,b]^{2^{\beta-1}};$$

the last equality since $2\beta - 1 \geq \beta$, and $[a,b,b]^{2^{\beta-1}}$ is of order 2. A straightforward calculation shows that we may choose the exponents in (5.6) so that r is unique modulo 2^α; s is unique modulo 2^β, t is unique modulo $2^{\beta+1}$; u is either unique modulo 2^β if $\alpha \neq \beta$, or satisfies $0 \leq u < 2^{\beta-1}$ if $\alpha = \beta$; and v satisfies $0 \leq v < 2^{\beta-1}$. Now simply note that $[a,b]^{2^\beta} \in N$, $[a,b]^{2^\beta} = [a,b,b]^{2^{\beta-1}}$ (and $[a,b]^{2^\beta} = [a,b,a]^{2^{\beta-1}}$ if $\alpha = \beta$), and we obtain the normal form described by taking the quotient. ☐

Corollary 5.7. *Let K be as in Theorem 5.5. Then $Z(K)$ is generated by a^{2^β}, $[a,b]^{2^\gamma}$, $[a,b,a]$, and $[a,b,b]$. Therefore,*

$$K/Z(K) \cong \left\langle x,y \;\middle|\; x^{2^\beta} = y^{2^\beta} = [x,y]^{2^\gamma} = [x,y,x] = [x,y,y] = e\right\rangle.$$

Proof. Using the normal form and Proposition 2.1, it is straightforward to verify that the elements given generate the center; the description of the quotient follows by mapping a to x and b to y. ☐

These corollaries, together with Prop. 2.2 and Theorem 2.3, yield the coproduct type case:

Corollary 5.8. *Let G be a two-generated 2-group of class two and of coproduct type; that is, presented as in Theorem 3.1(i). Then G is capable if and only if $\alpha = \beta$, or $\alpha = \beta + 1$ and $\gamma = \beta$.*

6. General type, $\gamma < \beta$

We turn now to the case of groups of general type that have $\gamma < \beta$; by Theorem 2.3, we may assume that $\alpha = \beta$.

Let K be the group described in Theorem 5.5. Let N be the subgroup of K generated by $[a^{2^{\alpha+\sigma-\gamma}}[a,b]^{-2^\sigma}, a]$ and $[a^{2^{\alpha+\sigma-\gamma}}[a,b]^{-2^\sigma}, b]$. We will show that K/N has G as a central quotient, provided $\alpha > \gamma + 1$. First we give a better description of N. We rewrite the elements above in normal form:

$$\begin{aligned}
[a^{2^{\alpha+\sigma-\gamma}}[a,b]^{-2^\sigma}, a] &= [[a,b]^{-2^\sigma}, a] = [a,b,a]^{-2^\sigma}.\\
[a^{2^{\alpha+\sigma-\gamma}}[a,b]^{-2^\sigma}, b] &= [a^{2^{\alpha+\sigma-\gamma}}, b][[a,b]^{-2^\sigma}, b]\\
&= [a,b]^{2^{\alpha+\sigma-\gamma}}[a,b,a]^{\binom{2^{\alpha+\sigma-\gamma}}{2}}[a,b,b]^{-2^\sigma}\\
&= [a,b]^{2^{\alpha+\sigma-\gamma}}[a,b,a]^{(2^{\alpha+\sigma-\gamma-1})(2^{\alpha+\sigma-\gamma}-1)}[a,b,b]^{-2^\sigma}.
\end{aligned}$$

Since $\alpha+\sigma \geq 2\gamma$, we must have $\alpha+\sigma-\gamma-1 \geq \gamma-1 \geq \sigma$. Thus, the subgroup N is generated by $[a,b]^{2^{\alpha+\sigma-\gamma}}[a,b,b]^{-2^\sigma}$ and $[a,b,a]^{2^\sigma}$. Since both elements are central,

we have that an element written in normal form $a^rb^s[a,b]^t[a,b,a]^u[a,b,b]^v\in K$ will lie in N if and only if $r\equiv s\equiv 0 \pmod{2^\alpha}$, $u\equiv v\equiv 0\pmod{2^\sigma}$, and $t+2^{\alpha-\gamma}v\equiv 0 \pmod{2^\alpha}$. Note that the last expression is well defined, since v is well defined modulo 2^γ.

If G is to be the central quotient of K/N, then we must have that $kN\in Z(K/N)$ if and only if k lies in $\langle a^{2^{\alpha+\sigma-\gamma}}[a,b]^{-2^\sigma}\rangle Z(K)$. Let $k\in K$ be an arbitrary element, written in normal form as $k=a^rb^s[a,b]^t[a,b,a]^u[a,b,b]^v$, and assume that kN is central in K/N. That means that both $[k,a]$ and $[k,b]$ lie in N. We have:

$$[k,a] = [a,b]^{-s}[a,b,a]^t[a,b,b]^{-\binom{s}{2}}. \tag{6.1}$$

$$[k,b] = [a,b]^r[a,b,a]^{\binom{r}{2}}[a,b,b]^{rs+t}. \tag{6.2}$$

Therefore, we must have:

$$\begin{aligned} -s-2^{\alpha-\gamma}\binom{s}{2} = -s\left(1+2^{\alpha-\gamma-1}(s-1)\right) &\equiv 0 \pmod{2^\alpha}\\ r+2^{\alpha-\gamma}(rs+t) &\equiv 0 \pmod{2^\alpha}\\ t\equiv\binom{s}{2}\equiv\binom{r}{2}\equiv rs+t &\equiv 0 \pmod{2^\sigma}. \end{aligned}$$

From the first two conditions we obtain that r and s are even. If $\alpha>\gamma+1$, then the first congruence gives $s\equiv 0\pmod{2^\alpha}$, and the second reduces to $r+2^{\alpha-\gamma}t\equiv 0 \pmod{2^\alpha}$. Since t is divisible by 2^σ, we have $t=m2^\sigma$ and $r\equiv -m2^{\alpha+\sigma-\gamma}\pmod{2^\alpha}$ for some integer m. The remaining congruences now follow. Thus:

$$k=\left(a^{2^{\alpha+\sigma-\gamma}}[a,b]^{2^\sigma}\right)^{-m}[a,b,a]^{u+\binom{-m}{2}2^{\alpha+2\sigma-\gamma}}[a,b,b]^v,$$

so k lies in the desired subgroup.

If $\alpha=\gamma+1$, then from $\alpha+\sigma\geq\gamma>\sigma$ we conclude that $\sigma+1=\gamma$; as far as our congruences are concerned, $r=t=0$ and $s=2^{\alpha-1}=2^\gamma$ is also a solution but does not fit into the subgroup we want. This suggests the following observation:

Lemma 6.3. *Let K be a nilpotent group of class 3, and let x and y be elements of K. Assume that $\alpha>1$ is an integer such that x^{2^α}, $[x,y]^{2^{\alpha-1}}$, and $x^{2^{\alpha-1}}[x,y]^{-2^{\alpha-2}}$ centralize $\langle x,y\rangle$. Then $y^{2^{\alpha-1}}$ commutes with x.*

Proof. Since $[x,y]^{2^{\alpha-1}}\in Z(\langle x,y\rangle)$, we must have $e=[x,y,x]^{2^{\alpha-1}}=[x,y,y]^{2^{\alpha-1}}$. Therefore, $[x,y,y]^{-2^{\alpha-2}}=[x,y,y]^{2^{\alpha-2}}$. In addition, since $[x,y]^{2^{\alpha-1}}$ centralizes $\langle x,y\rangle$, we have $[x,y]^{-2^{\alpha-2}}\equiv[x,y]^{2^{\alpha-2}}\pmod{Z(\langle x,y\rangle)}$.

From the other centralizing elements we deduce that:

$$e=[x^{2^\alpha},y]=[x,y]^{2^\alpha}[x,y,x]^{\binom{2^\alpha}{2}}=[x,y]^{2^\alpha}.$$

and also that $[x^{2^{\alpha-1}}[x,y]^{2^{\alpha-2}},x]=[x,y,x]^{2^{\alpha-2}}=e$. In addition,

$$\begin{aligned} e &= [x^{2^{\alpha-1}}[x,y]^{2^{\alpha-2}},y]=[x,y]^{2^{\alpha-1}}[x,y,x]^{\binom{2^{\alpha-1}}{2}}[x,y,y]^{2^{\alpha-2}}\\ &= [x,y]^{2^{\alpha-1}}[x,y,y]^{2^{\alpha-2}}=[x,y]^{2^{\alpha-1}}[x,y,y]^{-2^{\alpha-2}}. \end{aligned}$$

From these equations we obtain:

$$\begin{aligned} [x,y^{2^{\alpha-1}}] &= [x,y]^{2^{\alpha-1}}[x,y,y]^{\binom{2^{\alpha-1}}{2}}=[x,y]^{2^{\alpha-1}}[x,y,y]^{2^{\alpha-2}(2^{\alpha-1}-1)}\\ &= [x,y]^{2^{\alpha-1}}[x,y,y]^{-2^{\alpha-2}}[x,y,y]^{2^{2\alpha-3}}=e. \end{aligned}$$

The last equality holds since $\alpha > 1$, so $[x,y,y]^{2^{2\alpha-3}} = e$. $\square$

Thus we obtain:

Corollary 6.4. *Let G be a group of general type (that is, presented as in Theorem 3.1(ii)) with $\gamma < \beta$. Then G is capable if and only if $\alpha = \beta$ and $\gamma < \beta - 1$.*

Proof. We have seen the necessity of $\alpha = \beta$. If $\gamma + 1 < \beta = \alpha$, we have shown that the group K/N described above satisfies $(K/N)/Z(K/N) \cong G$, so G is capable. Assume then that $\gamma + 1 = \beta = \alpha$ (and so $\sigma = \gamma - 1 = \alpha - 2$), and let H be a group such that $H/N \cong G$, with $N \subset Z(H)$. We want to show that $N \neq Z(H)$. Note that H must be of class at most three. Let $x, y \in H$ project down to a and b, respectively. In H we have that x^{2^α}, $[x,y]^{2^\gamma} = [x,y]^{2^{\alpha-1}}$, and $x^{2^{\alpha+\sigma-\gamma}}[x,y]^{-2^\sigma} = x^{2^{\alpha-1}}[x,y]^{-2^{\alpha-2}}$ lie in N, and therefore are central. From the lemma we conclude that $y^{2^{\alpha-1}} \in Z(H)$. Since $b^{2^{\alpha-1}} \neq e$, we must have $N \neq Z(H)$, which proves G cannot be capable. $\square$

7. General type, $\gamma = \beta$

Although the description of groups of general type admits the case with $\alpha < \beta$, such a group cannot be capable: if $\alpha < \beta$, then Theorem 2.3 implies that $\gamma = \alpha$, and this makes $\alpha + \sigma \geq 2\gamma$ impossible. So we must have $\alpha \geq \beta$. And if $\gamma = \beta$, then we must have $\alpha > \beta$, for otherwise we again cannot satisfy the condition $\alpha + \sigma \geq 2\gamma$. Thus, the case we are to consider now is $\alpha = \beta + 1 = \gamma + 1$. This in turns implies $\sigma = \alpha - 2$. In this situation, $\alpha + \sigma - \gamma = \beta$.

Since $\gamma = \beta$, it is preferable to use the normal form in Theorem 5.2. Let a generate a cyclic group of order $2^\alpha = 2^{\beta+1}$, and let b generate a cyclic group of order 2^β. Let $K = \langle a \rangle \amalg^{\mathfrak{N}_3} \langle b \rangle$, and let N be the subgroup of K generated by the elements $[a^{2^\beta}[a,b]^{-2^{\beta-1}}, a]$ and $[a^{2^\beta}[a,b]^{-2^{\beta-1}}, b]$. We want to show that if $k \in K$ has $[k,a], [k,b] \in N$, then k lies in the subgroup $\langle a^{2^\beta}[a,b]^{-2^{\beta-1}}\rangle Z(K)$, and so deduce that G is the central quotient of K/N.

First, we calculate the normal forms of the generators of N:

$$\begin{aligned}
[a^{2^\beta}[a,b]^{-2^{\beta-1}}, a] &= [a,b,a]^{-2^{\beta-1}} = \left([a,b]^{-2}[a^2,b]\right)^{-2^{\beta-1}}\\
&= [a,b]^{2^\beta}[a^2,b]^{-2^{\beta-1}} = [a,b]^{\pm 2^\beta}[a^2,b]^{\pm 2^{\beta-1}}.\\
[a^{2^\beta}[a,b]^{-2^{\beta-1}}, b] &= [a,b]^{2^\beta}[a,b,a]^{\binom{2^\beta}{2}}[a,b,b]^{-2^{\beta-1}}\\
&= [a,b]^{2^\beta}\left([a,b]^{-2}[a^2,b]\right)^{\binom{2^\beta}{2}}\left([a,b]^{-2}[a,b^2]\right)^{-2^{\beta-1}}\\
&= [a,b]^{2^\beta - 2\binom{2^\beta}{2} + 2^\beta}[a^2,b]^{\binom{2^\beta}{2}}[a,b^2]^{-2^{\beta-1}}\\
&= [a,b]^{-2^\beta(2^\beta-1)}[a^2,b]^{2^{\beta-1}(2^\beta-1)}\\
&= [a,b]^{2^\beta}[a^2,b]^{-2^{\beta-1}} = [a,b]^{\pm 2^\beta}[a^2,b]^{\pm 2^{\beta-1}}.
\end{aligned}$$

That is, N is central and cyclic of order 2, generated by $[a,b]^{\pm 2^\beta}[a^2,b]^{\pm 2^{\beta-1}}$. The liberty in signs is because both $[a,b]^{2^\beta}$ and $[a^2,b]^{2^{\beta-1}}$ are of order two.

An arbitrary element $a^r b^s [a,b]^t [a^2,b]^u [a,b^2]^v$ will be in N if and only if $r \equiv 0 \pmod{2^{\beta+1}}$, $s \equiv 0 \pmod{2^\beta}$, $u \equiv v \equiv 0 \pmod{2^{\beta-1}}$ (note that $[a,b^2]$ is of order $2^{\beta-1}$, while $[a^2,b]$ is of order 2^β), and $t + 2u \equiv 0 \pmod{2^{\beta+1}}$. This last expression is well defined, since u is well defined modulo 2^β.

Now let $k\in K$ be given by $k=a^rb^s[a,b]^t[a^2,b]^u[a,b^2]^v$, and assume that both commutators $[k,a]$ and $[k,b]$ lie in N. We may start from (6.1) and (6.2), and substitute the values of $[a,b,a]$ and $[a,b,b]$. We obtain:

$$\begin{aligned}[k,a] &= [a,b]^{2\binom{s}{2}-s-2t}[a^2,b]^t[a,b^2]^{-\binom{s}{2}},\\ [k,b] &= [a,b]^{r-2\binom{r}{2}-2(rs+t)}[a^2,b]^{\binom{r}{2}}[a,b^2]^{rs+t}.\end{aligned}$$

Since both $[k,a]$ and $[k,b]$ lie in N, we must have r and s even; in addition, we have $\binom{s}{2}\equiv\binom{r}{2}\equiv 0 \pmod{2^{\beta-1}}$. We conclude that $r\equiv s\equiv 0\pmod{2^\beta}$. Since s is only defined modulo 2^β, we may take $s=0$.

Finally, we must also have

$$r-2\binom{r}{2}-2t+2\binom{r}{2}\equiv r-2t\equiv 0 \pmod{2^{\beta+1}}.$$

Since $r\equiv 0\pmod{2^\beta}$ and $t\equiv 0\pmod{2^{\beta-1}}$, we have the following possibilities for r and t: $r=t=0$; or $r=0$, $t=2^\beta$; or $r=2^\beta$, $t=\pm 2^{\beta-1}$; or $r=2^\beta$, $t=2^\beta\pm 2^{\beta-1}$.

If $r=0$, we obtain $k\in K_3\subset Z(K)$. If $r=2^\beta$, then we have k in the coset of $a^{2^\beta}[a,b]^{-2^{\beta-1}}$ modulo $Z(K)$. In any case, we conclude that kN is central in K/N, if and only if $k\in\langle a^{2^\beta}[a,b]^{-2^{\beta-1}}\rangle Z(K)$. So G is the central quotient of K/N.

Corollary 7.1. *Let G be a group of general type (that is, presented as in Theorem 3.1(ii)) with $\gamma=\beta$. Then G is capable if and only if $\alpha=\beta+1$ and $\sigma=\beta-1$.*

8. Conclusion

We summarize our results in the following theorem:

Theorem 8.1. *Let G be a 2-generated 2-group of class two, presented as in Theorem 3.1. Then G is capable if and only if one of the following hold:*

(a) *G is of type (i) and $\alpha=\beta$; or*
(b) *G is of type (i), and $\alpha=\beta+1=\gamma+1$; or*
(c) *G is of type (ii), $\alpha=\beta$, and $\gamma<\beta-1$; or*
(d) *G is of type (ii), and $\alpha=\beta+1=\gamma+1=\sigma+2$.*

In particular, if G is of type (iii) (exceptional type), then G is not capable.

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